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Sound Synthesis for Contact-Driven Musical Instruments via Discretisation of Hamilton's Equations

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Physical modelling of musical instruments involves studying nonlinear interactions between parts of the instrument. These can pose several difficulties concerning the accuracy and stability of numerical algorithms. In particular, when the underlying forces are non-analytic functions of the phase-space variables, a stability proof can only be obtained in limited cases. An approach has been recently presented by the authors, leading to unconditionally stable simulations for lumped collision models. In that study, discretisation of Hamilton's equations instead of the usual Newton's equation of motion yields a numerical scheme that can be proven to be energy conserving. In this paper, the above approach is extended to collisions of distributed objects. Namely, the interaction of an ideal string with a flat barrier is considered. The problem is formulated within the Hamiltonian framework and subsequently discretised. The resulting nonlinear matrix equation can be shown to possess a unique solution, that enables the update of the algorithm. Energy conservation and thus numerical stability follows in a way similar to the lumped collision model. The existence of an analytic description of this interaction allows the validation of the model's accuracy. The proposed methodology can be used in sound synthesis applications involving musical instruments where collisions occur either in a confined (e.g. hammer-string interaction, mallet impact) or in a distributed region (e.g. string-bridge or reed-mouthpiece interaction).

1 Introduction

Nonlinear phenomena inherently occur during sound generation by musical instruments. Physics-based sound synthesis approaches need to take these nonlinearities into account, in order to produce faithful simulations. The presence of nonlinear terms in the equations that describe the instrument oscillations pose several difficulties to both analytical and numerical approaches [1]. One common cause of nonlinear effects is the collision between parts of the instrument. Such collisions can occur in either a lumped or a distributed manner. In the field of music acoustics, collisions are usually modelled using a penalty approach [2], with the collision force given by

$$f(\Delta y) = k_b [\Delta y^\alpha] \quad \text{where} \quad [\Delta y] = h(\Delta y) \Delta y. \quad (1)$$

h is the Heaviside step function, Δy the compression along the displacement axis y and the force f is active only for positive compression values. k_b and α are power-law constants depending on the nature of the barrier. Several time-stepping methods have been used for the simulation of collisions, most of which are based on finite differences [3, 4] or closely related methods such as the trapezoidal rule [5] or Verlet integration [6]. Proving the stability of the corresponding numerical schemes requires the use of energy methods [4, 7] due to the presence of nonlinear terms. Energy conserving schemes can thus be formulated for the simulation of conservative, nonlinear systems [8, 9]. However, the definition of a numerical energy invariant is not as straightforward for systems involving collision forces, which are non-analytic functions of y . Describing the system using Hamilton's equations and discretising these instead of Newton's equation of motion yields numerical schemes that can be shown to conserve energy. This approach has been recently presented by the authors for the case of lumped collisions [10]. This paper extends it to distributed systems, studying the vibration of a string interacting with a rigid barrier.

In Section 2 the equations of motion for the vibrating string are derived from a variational principle. In Section 3 a numerical scheme is formulated based on the above energetic approach, and the stability and numerical solution of the derived scheme is discussed. Section 4 presents simulations of a vibrating string colliding with a rigid barrier and Section 5 evaluates the proposed methodology within the context of sound synthesis.

2 Lagrangian Formulation

Let a stiff string of length l , simply supported at both ends and with given initial displacement $y(x)$ interact with a flat, rigid barrier located below it at height y_b . The Lagrangian density of this system is given by the difference between the kinetic energy density

$$\mathcal{T} = \rho A (\partial_t y)^2 / 2 \quad (2)$$

and the potential energy density

$$\mathcal{V} = \mathcal{V}_\tau + \mathcal{V}_s + \mathcal{V}_b \quad (3)$$

where ρ is the mass density and A the cross-sectional area of the string,

$$\mathcal{V}_\tau = \tau (\partial_x y)^2 / 2 \quad (4)$$

the potential energy due to the string tension τ ,

$$\mathcal{V}_s = EI (\partial_{xx} y)^2 / 2 \quad (5)$$

the potential energy due to string stiffness EI and

$$\mathcal{V}_b = k_b [(y_b - y)^{\alpha+1}] / (\alpha + 1) \quad (6)$$

the collision potential due to interaction with the barrier. Hence the Lagrangian density $\mathcal{L} = \mathcal{T} - \mathcal{V}$ is a function of the displacement variable $y(x, t)$ and its space-time derivatives

$$\mathcal{L} = \frac{\rho A}{2} y_t^2 - \frac{\tau}{2} y_x^2 - \frac{EI}{2} y_{xx}^2 - \frac{k_b [(y_b - y)^{\alpha+1}]}{\alpha + 1} \quad (7)$$

and the Lagrangian of the system is given by

$$L = \int_0^l \mathcal{L}(y, y_t, y_x, y_{xx}; x, t) dx \quad (8)$$

where the following notation is adopted:

$$y_t = \partial_t y = \partial y / \partial t \quad (9a)$$

$$y_x = \partial_x y = \partial y / \partial x \quad (9b)$$

$$y_{xx} = \partial_{xx} y = \partial^2 y / \partial x^2. \quad (9c)$$

The variation of the Lagrangian density subject to a virtual displacement δy is

$$\delta \mathcal{L} = \delta y \frac{\partial \mathcal{L}}{\partial y} + \frac{\partial \delta y}{\partial t} \frac{\partial \mathcal{L}}{\partial y_t} + \frac{\partial \delta y}{\partial x} \frac{\partial \mathcal{L}}{\partial y_x} + \frac{\partial^2 \delta y}{\partial x^2} \frac{\partial \mathcal{L}}{\partial y_{xx}} \quad (10)$$

and Hamilton's principle of least action [11] dictates that

$$\delta \int L dt = 0 \quad \Rightarrow \quad \delta \iint \mathcal{L} dx dt = 0. \quad (11)$$

Substituting the expression in Eq. (10) and using integration by parts, along with the fact that δy vanishes at the integration boundaries [12], yields

$$\iint \delta y \left(\frac{\partial \mathcal{L}}{\partial y} - \frac{\partial}{\partial t} \frac{\partial \mathcal{L}}{\partial y_t} - \frac{\partial}{\partial x} \frac{\partial \mathcal{L}}{\partial y_x} + \frac{\partial^2}{\partial x^2} \frac{\partial \mathcal{L}}{\partial y_{xx}} \right) dx dt = 0. \quad (12)$$

The requirement for the integral to be zero for an arbitrary variation δy results in the Euler-Lagrange equation for the given dynamical system

$$\frac{\partial \mathcal{L}}{\partial y} = \frac{\partial}{\partial t} \left(\frac{\partial \mathcal{L}}{\partial y_t} \right) + \frac{\partial}{\partial x} \left(\frac{\partial \mathcal{L}}{\partial y_x} \right) - \frac{\partial^2}{\partial x^2} \left(\frac{\partial \mathcal{L}}{\partial y_{xx}} \right). \quad (13)$$

2.1 Hamilton's equations

The Hamiltonian density \mathcal{H} can be obtained by defining the conjugate momentum

$$p = \frac{\partial \mathcal{L}}{\partial y_t} = \rho A y_t \quad (14)$$

and taking the Legendre transformation of the Lagrangian density

$$\mathcal{H} = y_t p - \mathcal{L}(y, y_t, y_x, y_{xx}) \quad (15a)$$

$$= \frac{1}{2} \frac{p^2}{\rho A} + \frac{1}{2} \tau y_x^2 + \frac{1}{2} E I y_{xx}^2 + \frac{k_b}{\alpha + 1} [(y_b - y)^{\alpha+1}] \quad (15b)$$

$$= \mathcal{T}(p) + \mathcal{V}_\tau(y_x) + \mathcal{V}_s(y_{xx}) + \mathcal{V}_b(y). \quad (15c)$$

Equations (13), (14) and (15a) can be combined to formulate Hamilton's equations of motion:

$$\frac{\partial p}{\partial t} = \frac{\partial}{\partial x} \left(\frac{\partial \mathcal{H}}{\partial y_x} \right) - \frac{\partial^2}{\partial x^2} \left(\frac{\partial \mathcal{H}}{\partial y_{xx}} \right) - \frac{\partial \mathcal{H}}{\partial y} \quad (16a)$$

$$\frac{\partial y}{\partial t} = \frac{\partial \mathcal{H}}{\partial p} \quad (16b)$$

and the Hamiltonian (total energy) of the system is computed as

$$H = \int_0^L \mathcal{H}(y, p, y_x, y_{xx}) dx. \quad (17)$$

3 Numerical Formulation

For simplicity of presenting the numerical model, and in order to compare with existing analytic solutions, the discretisation of Hamilton's equations will be performed on an ideal (flexible) string; the string stiffness is neglected, resulting in $\mathcal{V}_s = 0$. Mid-point derivative approximations are employed to derive a numerical scheme, where y_m^n denotes the value of variable y at position $x = m\Delta x$ and time $t = n\Delta t$, Δx being the spatial sampling interval and $f_s = 1/\Delta t$ the sampling rate (see Fig. 1). Rewriting (16) using Eq. (15c), Hamilton's equations are approximated by

$$\frac{p_m^{n+1} - p_m^n}{\Delta t} = \frac{\left\{ \frac{\partial \mathcal{V}_\tau}{\partial y_x} \right\}_{m+\frac{1}{2}}^{n+\frac{1}{2}} - \left\{ \frac{\partial \mathcal{V}_\tau}{\partial y_x} \right\}_{m-\frac{1}{2}}^{n+\frac{1}{2}}}{\Delta x} - \frac{\mathcal{V}_b(y_m^{n+1}) - \mathcal{V}_b(y_m^n)}{y_m^{n+1} - y_m^n} \quad (18a)$$

$$\frac{y_m^{n+1} - y_m^n}{\Delta t} = \frac{\mathcal{T}(p_m^{n+1}) - \mathcal{T}(p_m^n)}{p_m^{n+1} - p_m^n} \quad (18b)$$

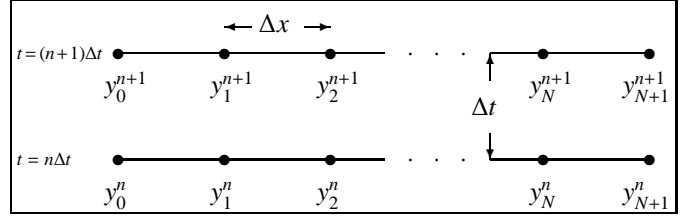


Figure 1: The discretised string at times $n\Delta t$ and $(n+1)\Delta t$.

where, for example,

$$\left\{ \frac{\partial \mathcal{V}_\tau}{\partial y_x} \right\}_{m+\frac{1}{2}}^{n+\frac{1}{2}} = \frac{\mathcal{V}_\tau(y_{x_{m+\frac{1}{2}}}^{n+1}) - \mathcal{V}_\tau(y_{x_{m+\frac{1}{2}}}^n)}{y_{x_{m+\frac{1}{2}}}^{n+1} - y_{x_{m+\frac{1}{2}}}^n} \quad (19)$$

approximates the partial derivative of \mathcal{V}_τ with respect to y_x .

It is useful here to introduce the forward and backward space shift operators, through their action on y_m^n , as

$$\delta_+ y_m^n = \frac{y_{m+1}^n - y_m^n}{\Delta x}, \quad \delta_- y_m^n = \frac{y_m^n - y_{m-1}^n}{\Delta x}. \quad (20)$$

Using the following approximations for y_x

$$y_{x_{m+\frac{1}{2}}}^n = \delta_+ y_m^n, \quad y_{x_{m-\frac{1}{2}}}^n = \delta_- y_m^n \quad (21)$$

a scheme centred at time $t = (n+1/2)\Delta t$ and position $x = m\Delta x$ is obtained

$$\frac{p_m^{n+1} - p_m^n}{\Delta t} = \frac{\tau}{2} \delta_\Delta (y_m^{n+1} + y_m^n) - \frac{k_b}{\alpha + 1} \frac{[(y_b - y_m^{n+1})^{\alpha+1}] - [(y_b - y_m^n)^{\alpha+1}]}{y_m^{n+1} - y_m^n} \quad (22a)$$

$$\frac{y_m^{n+1} - y_m^n}{\Delta t} = \frac{1}{\rho A} \frac{p_m^{n+1} + p_m^n}{2} \quad (22b)$$

where $\delta_\Delta = (\delta_+ \delta_-)$. In matrix form this can be written as

$$\mathbf{p}^{n+1} - \mathbf{p}^n = \phi \mathbf{D}_2 (\mathbf{y}^{n+1} + \mathbf{y}^n) - \varpi \mathbf{S}^{-1} ([(\mathbf{y}_b - \mathbf{y}^{n+1})^{\alpha+1}] - [(\mathbf{y}_b - \mathbf{y}^n)^{\alpha+1}]) \quad (23a)$$

$$\theta (\mathbf{y}^{n+1} - \mathbf{y}^n) = \mathbf{p}^{n+1} + \mathbf{p}^n. \quad (23b)$$

where $\mathbf{S} = \text{diag}(\mathbf{y}^{n+1} - \mathbf{y}^n)$ is a diagonal matrix,

$$\phi = \frac{\tau \Delta t}{2 \Delta x^2}, \quad \varpi = \frac{k_b \Delta t}{\alpha + 1}, \quad \theta = \frac{2 \rho A}{\Delta t} \quad (24)$$

and \mathbf{y}^n , \mathbf{y}_b^n and \mathbf{p}^n are column vectors holding displacement, barrier profile and momentum values. Under the assumption of simply supported boundary conditions on both ends of the system, these vectors hold the values of N interior nodes on the string (i.e. from y_1 to y_N), and \mathbf{D}_2 then is an $N \times N$ tridiagonal matrix:

$$\mathbf{D}_2 = \begin{bmatrix} -2 & 1 & & 0 \\ 1 & \ddots & \ddots & \\ & \ddots & \ddots & 1 \\ 0 & & 1 & -2 \end{bmatrix} \quad (25)$$

which implements the second spatial derivative of the string state. It is convenient to rewrite the scheme using a scaled

momentum variable $\mathbf{q}^n = \mathbf{p}^n/\theta$:

$$\mathbf{q}^{n+1} - \mathbf{q}^n = \beta_2 \mathbf{D}_2 (\mathbf{y}^{n+1} + \mathbf{y}^n) \quad (26a)$$

$$- \zeta \mathbf{S}^{-1} \left(\lfloor (\mathbf{y}_b - \mathbf{y}^{n+1})^{\alpha+1} \rfloor - \lfloor (\mathbf{y}_b - \mathbf{y}^n)^{\alpha+1} \rfloor \right)$$

$$\mathbf{y}^{n+1} - \mathbf{y}^n = \mathbf{q}^{n+1} + \mathbf{q}^n \quad (26b)$$

where $\beta_2 = \phi/\theta$ and $\zeta = \varpi/\theta$. Now setting

$$\mathbf{s} = \mathbf{y}^{n+1} - \mathbf{y}^n = \mathbf{q}^{n+1} + \mathbf{q}^n \quad (27)$$

yields the nonlinear system of equations

$$\begin{aligned} \mathbf{F} &= (\mathbf{I} - \beta_2 \mathbf{D}_2) \mathbf{s} - 2(\beta_2 \mathbf{D}_2 \mathbf{y}^n + \mathbf{q}^n) \\ &+ \zeta \mathbf{S}^{-1} \left(\lfloor (\mathbf{y}_b - \mathbf{y}^n - \mathbf{s})^{\alpha+1} \rfloor - \lfloor (\mathbf{y}_b - \mathbf{y}^n)^{\alpha+1} \rfloor \right) = \mathbf{0}. \end{aligned} \quad (28)$$

3.1 Conservation of energy

The total energy of the system can be calculated by integrating the energy densities along the length of the string, i.e.

$$H^n = \sum_{m=0}^{N+1} (\mathcal{T}_m^n + \mathcal{V}_m^n) \Delta x \quad (29)$$

where $\mathcal{T}_m^n = \mathcal{T}(\mathbf{p}_m^n)$ and $\mathcal{V}_m^n = \frac{\mathcal{V}_\tau(\delta_- \mathbf{y}_m^n) + \mathcal{V}_\tau(\delta_+ \mathbf{y}_m^n)}{2} + \mathcal{V}_b(\mathbf{y}_m^n)$ represent kinetic and potential energy densities, respectively. In matrix form the numerical Hamiltonian is given by

$$H^n = b[\mathbf{q}^t \mathbf{q} - \beta_2 \mathbf{y}^t \mathbf{D}_2 \mathbf{y} + \zeta \mathbf{1}^t \lfloor (\mathbf{y}_b - \mathbf{y})^{\alpha+1} \rfloor] \quad (30)$$

with $\mathbf{1} = (1, \dots, 1)^t$ and $b = 2\rho A \Delta x / \Delta t^2$. Multiplying the left hand side of Eq. (26a) with $(\mathbf{q}^{n+1} + \mathbf{q}^n)^t$ and the right hand side with $(\mathbf{y}^{n+1} - \mathbf{y}^n)^t$, which are equivalent by (26b), yields

$$\begin{aligned} (\mathbf{q}^{n+1} + \mathbf{q}^n)^t (\mathbf{q}^{n+1} + \mathbf{q}^n) &= \beta_2 (\mathbf{y}^{n+1} - \mathbf{y}^n)^t \mathbf{D}_2 (\mathbf{y}^{n+1} + \mathbf{y}^n) \\ &- \zeta (\mathbf{y}^{n+1} - \mathbf{y}^n)^t \mathbf{S}^{-1} \left(\lfloor (\mathbf{y}_b - \mathbf{y}^{n+1})^{\alpha+1} \rfloor - \lfloor (\mathbf{y}_b - \mathbf{y}^n)^{\alpha+1} \rfloor \right) \end{aligned} \quad (31)$$

which, given that \mathbf{D}_2 is symmetric, can be written as

$$\begin{aligned} (\mathbf{q}^{n+1})^t \mathbf{q}^{n+1} - \beta_2 (\mathbf{y}^{n+1})^t \mathbf{D}_2 \mathbf{y}^{n+1} &+ \zeta \mathbf{1}^t \lfloor (\mathbf{y}_b - \mathbf{y}^{n+1})^{\alpha+1} \rfloor \\ &= (\mathbf{q}^n)^t \mathbf{q}^n - \beta_2 (\mathbf{y}^n)^t \mathbf{D}_2 \mathbf{y}^n + \zeta \mathbf{1}^t \lfloor (\mathbf{y}_b - \mathbf{y}^n)^{\alpha+1} \rfloor. \end{aligned} \quad (32)$$

Now multiplying by b and using the definition of the numerical energy in (30), it follows that

$$H^{n+1} = H^n \quad (33)$$

and the total energy of the system is conserved.

3.2 Numerical solution

Equation (28) can be solved for \mathbf{s} using the multidimensional Newton method, which requires forming the Jacobian of \mathbf{F}

$$\mathbf{J} = \mathbf{I} - \beta_2 \mathbf{D}_2 + \mathbf{C} \quad (34)$$

where \mathbf{C} is a diagonal matrix with elements

$$\{c_{i,i}\} = \frac{\Delta t}{\theta} \frac{s_i \mathcal{V}'_b(\mathbf{y}_i^n + s_i) - \mathcal{V}_b(\mathbf{y}_i^n + s_i) + \mathcal{V}_b(\mathbf{y}_i^n)}{s_i^2} \quad (35)$$

which, in accordance with the case of lumped collisions [10], can be shown to be positive definite. Here \mathcal{V}'_b signifies taking the derivative of \mathcal{V}_b with respect to displacement.

From the energy expression (30) it follows that $-\beta_2 \mathbf{D}_2$ is also positive definite hence \mathbf{J} is positive definite, which ensures the uniqueness of a root of Eq. (28) [13]. Singularities in both \mathbf{F} and its Jacobian can be handled as in the case of lumped collisions presented in [10], by using the limits of \mathbf{F} and \mathbf{J} when $s \rightarrow 0$. The update equation is $\mathbf{s} = \mathbf{s} - \mathbf{J}^{-1} \mathbf{F}$, where instead of forming the inverse matrix it is more efficient to solve a tridiagonal system. Global convergence of the Newton method is guaranteed for the componentwise convex function \mathbf{F} , since the Jacobian is an M -matrix [14]. Convergence is typically achieved in fewer than 20 iteration steps, by using the previous value of \mathbf{s} as a starting point.

4 Numerical Simulations

The simulation of an ideal string bouncing on a flat barrier is depicted in Figure 2. The top plot shows the displacement of the mid-point of the string. To verify the correct behaviour of the model, reference is made to an analytical result that compares the frequency of an impeded string to that of a free, flexible vibrating string [15]. This states that if a straight obstacle is placed halfway across the amplitude of the string vibration, the period of the impeded string will be 1.5 times the period of the free vibrating string [16]. This result is reproduced by a numerical simulation using scheme (26) for a 0.7 m long string, setting $k_b = 10^7$ in order to simulate a rigid obstacle with $\alpha = 1$. For comparison, the simulation was repeated with 20 times oversampling and using $k_b = 10^9$, which results in a very close approximation to the theoretical period ratio.

The bottom plot in Figure 2 shows the error in the conservation of the numerical energy of the system. Despite the proof of conservation of energy the Hamiltonian can be preserved only to machine precision in implementations on digital processors, due to quantisation in finite-precision arithmetic. The resulting energy error, expressed in terms of the deviation of $H^n = H(\mathbf{y}^n, \mathbf{q}^n)$ from the initial energy H^0 , reads (in normalized form)

$$e^n = (H^n - H^0)/H^0. \quad (36)$$

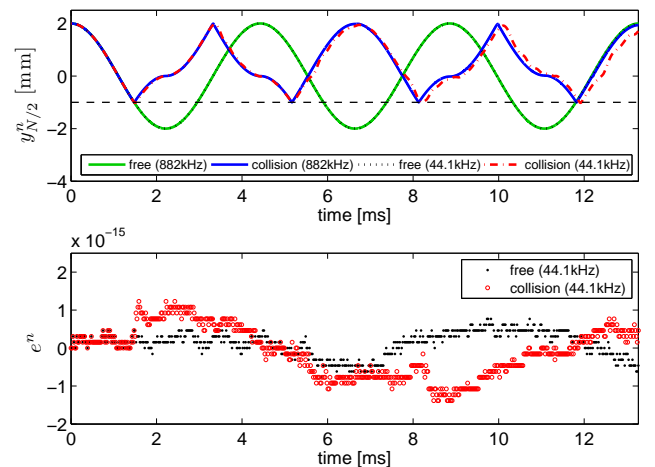


Figure 2: Simulation of an ideal string being free to vibrate or bouncing on a rigid obstacle, for the initial condition $y(x, 0) = 0.002 \sin(\pi x/l)$, with $\rho A = 0.001$ kg/m, $\tau = 100$ N and $\Delta x = 0.007$ m. Top: mid-point string displacement. Bottom: numerical energy error.

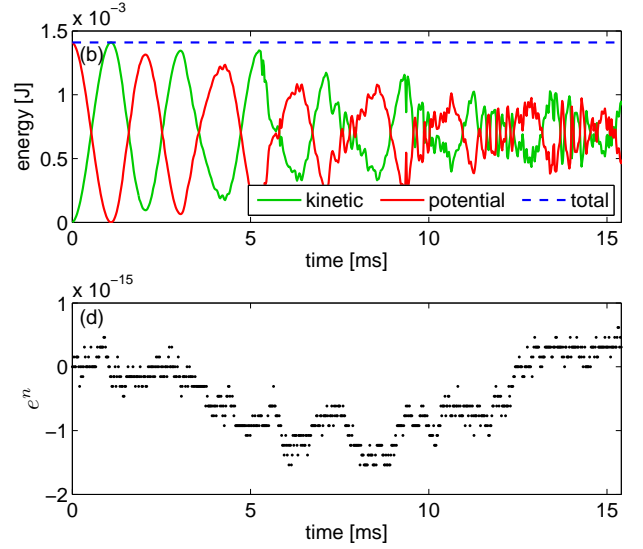
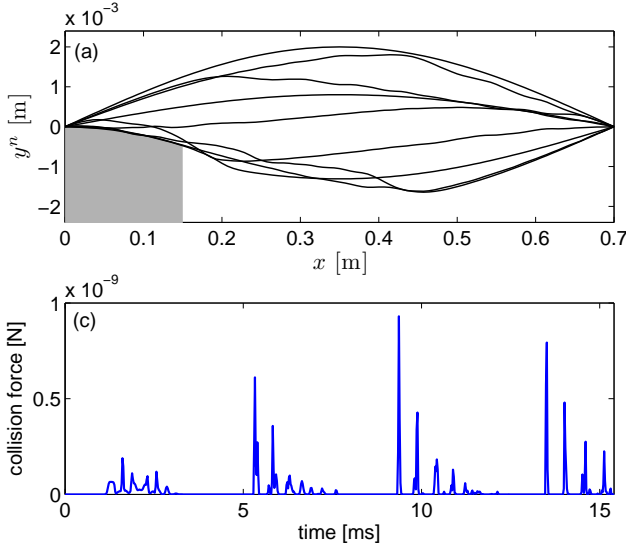


Figure 3: (a) Snapshots of a flexible string bouncing on a curved obstacle simulated using $f_s = 44.1$ kHz, (b) the energy components (c) the collision force due to string-obstacle interaction and (d) the energy error e^n .

It is worth noting here that quantisation generally results into a random-like signal e^n which, if zero mean, will not cause an energy shift over time.

A more interesting setting for music acoustics and sound synthesis applications is that of a string interacting with a curved barrier located at one of its ends. Such configurations can be encountered in some Asian instruments, where the string terminates on a relatively flat bridge. Fig. 3(a) shows the vibration of a flexible string in a similar setting. The string properties are taken from the previous experiment and a 15 mm long, curved barrier with profile $y_b = -0.01x^2/l^2$ is rigidly placed below the string. The contact of the string with this surface is characterised by $k_b = 10^{13}$ and $\alpha = 2.5$. The nonlinear behaviour of this system can be observed in the irregular exchange between the kinetic and the potential energy in Fig. 3(b). The total energy of the system remains constant to machine precision, as shown in Fig. 3(d). Fig. 3(c) reveals that the interaction between the string and the barrier involves multiple impacts during contact periods, which result in the generation of high-frequency vibrations that are characteristic of string instruments with flat bridges. For example, the buzzing sound of a sitar is understood to stem from such multiple impacts [17].

4.1 Application to a stiff, lossy string

In the previous section, the stiffness of the string has been neglected in order to simplify the discretisation process. However, precise modelling of stringed instruments requires one to include the effect of string stiffness. This involves higher order derivatives in Eq. (16), due to a nonzero potential $\mathcal{V}_s(y_{xx})$. Furthermore, in order to produce realistic sounds the damping of the string needs to be considered. It has been neglected so far, since the stability of the derived scheme has been established by virtue of the conservation of energy. Internal damping (η), due to the nature of the vibrating string, and external damping (γ), due to interaction with the surrounding fluid, can be included in the model using Kelvin-Voigt and resistive terms, as outlined in [3].

Using a discretisation procedure similar to that explained in Section 3 for the ideal string and including both damping

and string stiffness, leads to the following equations

$$\mathbf{q}^{n+1} - \mathbf{q}^n = -\mathbf{D} \left[\left(1 + \frac{2\eta}{\Delta t} \right) \mathbf{y}^{n+1} + \left(1 - \frac{2\eta}{\Delta t} \right) \mathbf{y}^n \right] - \frac{\gamma \Delta t}{2} (\mathbf{y}^{n+1} - \mathbf{y}^n) - \zeta \mathbf{S}^{-1} \left(\lfloor (\mathbf{y}_b - \mathbf{y}^{n+1})^{\alpha+1} \rfloor - \lfloor (\mathbf{y}_b - \mathbf{y}^n)^{\alpha+1} \rfloor \right) \quad (37a)$$

$$\mathbf{y}^{n+1} - \mathbf{y}^n = \mathbf{q}^{n+1} + \mathbf{q}^n \quad (37b)$$

with $\mathbf{D} = \beta_4 \mathbf{D}_4 - \beta_2 \mathbf{D}_2$, where $\beta_4 = EI \Delta t / (2\theta \Delta x^4)$ and $\mathbf{D}_4 = \mathbf{D}_2 \mathbf{D}_2$. As previously, the system is solved by finding the root s of a nonlinear function

$$\mathbf{F}_1 = \left[\left(1 + \frac{\gamma \Delta t}{2} \right) \mathbf{I} + \left(1 + \frac{2\eta}{\Delta t} \right) \mathbf{D} \right] s + 2(\mathbf{D} \mathbf{y}^n - \mathbf{q}^n) + \zeta \mathbf{S}^{-1} \left(\lfloor (\mathbf{y}_b - \mathbf{y}^n - s)^{\alpha+1} \rfloor - \lfloor (\mathbf{y}_b - \mathbf{y}^n)^{\alpha+1} \rfloor \right). \quad (38)$$

An example of such a stiff system is given in Figure 4, using $EI = 0.012 \text{ Nm}^2$, $\gamma = 100 \text{ s}^{-1}$ and $\eta = 10^{-5} \text{ s}$, the remaining parameters being identical to the case of the flexible string. A damped system behaviour is confirmed and, even though an energy preservation check does not apply, stability may be observed in that $\partial H / \partial t \leq 0$ at all times.

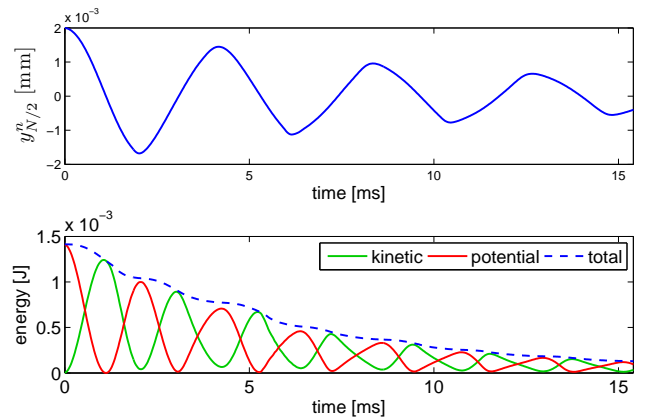


Figure 4: Simulation of a stiff, lossy string bouncing on a curved obstacle. Top: mid-point string displacement. Bottom: the energy components.

5 Conclusions

A method has been proposed to derive stable algorithms for the simulation of nonlinear distributed systems. The discretisation of the system is carried out after the model equations are derived using a variational approach. Deriving the numerical scheme starting from Hamilton's equations, rather than Newton's equations of motion, leads to an unconditional stability, which can be shown using the conservation of the system energy. The update of the scheme requires the solution of a set of nonlinear equations that can be shown to possess a unique root. Furthermore, global convergence of the Newton method can be established for the numerical solution of these equations.

The proposed methodology can be used to formulate time-domain models with improved robustness in comparison to methods previously applied to the simulation of contact-driven musical instruments. That is, unlike schemes derived by discretising a Newtonian description, the simulations do not suffer from energy jumps during the decoupling of the impacting objects. As a result, implementations require neither the energy corrections employed in various other formulations [6, 18] nor the monitoring of energy flows common to wave digital filter models [19]. In comparison to the digital waveguide model used in [20], the proposed approach has the advantage that the elasticity and damping properties of the contact can be specified. Damping phenomena during the impact may also be modelled, e.g. as proposed by Hunt and Crossley [21]. Such a formulation can be directly included in the proposed numerical scheme, as outlined elsewhere in these proceedings [22]. This may be necessary for the simulation of systems, where the nature of the impactive interaction is critical for the sound generation, such as in a tanpura or a sitar. Other impactive interactions of musical interest that the proposed schemes can be directly applied to include reed beating in woodwind instruments [23], string-bridge coupling in pianos [20], and braypin-string collisions in early harps.

References

- [1] A. Chaigne. Recent advances in vibration and radiation of musical instruments. *Flow, Turbulence and Combustion*, 61(1-4):31–41, 1998.
- [2] D. Hall. Piano string excitation. VI: Nonlinear modeling. *J. Acoust. Soc. Am.*, 92(1):95–105, 1992.
- [3] A. Chaigne and V. Doutaut. Numerical simulation of xylophones. I. Time-domain modeling of the vibrating bars. *J. Acoust. Soc. Am.*, 101(1):539–557, 1997.
- [4] S. Bilbao. *Numerical Sound Synthesis*. Wiley & Sons, Chichester, UK, 2009.
- [5] F. Avanzini and D. Rocchesso. Physical modeling of impacts: theory and experiments on contact time and spectral centroid. In *Proc. Sound and Music Computing*, pages 287–293, 2004.
- [6] S. Papetti, F. Avanzini, and D. Rocchesso. Numerical methods for a nonlinear impact model: A Comparative study with closed-form corrections. *IEEE Trans. Audio, Speech, Language Process.*, 19(7):2146–2158, 2011.
- [7] R. Richtmyer and K. Morton. *Difference Methods for Initial-Value Problems*. Wiley, New York, 1967.
- [8] S. Bilbao. Time domain simulation and sound synthesis for the snare drum. *J. Acoust. Soc. Am.*, 131(1):914–925, 2012.
- [9] J. Chabassier, A. Chaigne, and P. Joly. Modeling and simulation of a grand piano. *J. Acoust. Soc. Am.*, 134(1):648–665, 2013.
- [10] V. Chatziioannou and M. van Walstijn. An energy conserving finite difference scheme for simulation of collisions. In *Proc. Sound and Music Computing (SMAC-SMC 2013)*, pages 584–591, Stockholm, 2013.
- [11] V. Arnold. *Mathematical methods of classical mechanics*, volume 60. Springer, New York, 1978.
- [12] C. Lanczos. *The variational principles of mechanics*, volume 4. Courier Dover Publications, 1970.
- [13] P. Deuffhard. *Newton methods for nonlinear problems: Affine invariance and adaptive algorithms*. Springer, Berlin, 2004.
- [14] J. Ortega and W. Rheinboldt. *Iterative Solution of Nonlinear Equations in Several Variables*, volume 30. SIAM, New York, 1970.
- [15] A. Haraux and H. Cabannes. Almost periodic motion of a string vibrating against a straight fixed obstacle. *Nonlinear Analysis, Theory, Methods and Applications*, 7:129–141, 1983.
- [16] S. Han and M. Grosenbaugh. Non-linear free vibration of a cable against a straight obstacle. *Journal of Sound and Vibration*, 273:337–361, 2004.
- [17] C. Vyasarayani, S. Birkett, and J. McPhee. Modeling the dynamics of a vibrating string with a finite distributed unilateral constraint: Application to the sitar. *J. Acoust. Soc. Am.*, 125(6):3673–3682, 2009.
- [18] L. Trautmann and R. Rabenstein. Multirate simulations of string vibrations including nonlinear fret-string interactions using the functional transformation method. *Appl. Signal Processing*, (7):949–963, 2004.
- [19] A. Sarti and G. De Sanctis. Systematic methods for the implementation of nonlinear wave-digital structures. *IEEE Trans. Circuits and Systems I: Regular Papers*, 56(2):460–472, 2009.
- [20] A. Stulov and D. Kartofelev. Vibration of strings with nonlinear supports. *Appl. Acoustics*, 76:223–229, 2014.
- [21] K. Hunt and F. Crossley. Coefficient of restitution interpreted as damping in vibroimpact. *Journal of Applied Mechanics*, 42:440–445, 1975.
- [22] M. van Walstijn and V. Chatziioannou. Numerical simulation of tanpura string vibrations. In *Proc. International Symposium on Musical Acoustics*, Le Mans, 2014.
- [23] V. Chatziioannou and M. van Walstijn. A Hamiltonian approach to simulation of acoustic systems involving nonlinear interactions. *J. Acoust. Soc. Am.*, 134(5):4219, 2013.